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Note

A refinement of Vizing's theorem

Eckhard Steffen *

Princeton University Program in Applied and Computational Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA

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Abstract

Let M be a multigraph with maximum edge-multiplicity μ and girth g. We show that $\chi'(M) \leqslant \Delta(M) + \lceil \mu/\lfloor g/2 \rfloor \rceil$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we consider multigraphs $M = (V, E)$ with vertex set V and edge set E. Each edge is incident to precisely two different vertices, i.e. we do not allow loops, and $\mu(v, w)$ denotes the number of edges between the vertices v and w. The maximum vertex degree in M is denoted by $\Delta(M)$ and $\mu(M)$ denotes the maximum of $\{\mu(v, w) | v, w \in V\}$. The chromatic index $\chi'(M)$ is the minimum number of colors needed to color the edges of M such that adjacent edges are colored differently. A multigraph M is *critical*, if $\chi'(M) > \Delta(M)$ and $\chi'(M - e) < \chi'(M)$ for each edge $e \in E$.

Let M be a multigraph. Vizing [2] proved that $\chi'(M) \leqslant \Delta(M) + \mu(M)$. We will prove a refinement of this theorem with respect to the girth of M, where the *girth* of a multigraph is defined to be the girth of its underlying simple graph.

For the proof we will need some techniques developed by Kierstead [1]. Let $M = (V, E)$ be a critical multigraph, $e_0 \in E$, and ϕ a $(\chi'(M) - 1)$-coloring of $M - e_0$. We may consider ϕ as a $(\chi'(M) - 1)$-coloring of M leaving the edge e_0 uncolored. The set of colors not present at vertex v is denoted by $\bar{\phi}(v)$. A path P in M with distinct

* Correspondence address. Fakultät fur Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany.

E-mail address: steffen@math.princeton.edu (E. Steffen).

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vertices $v_0, v_1, \dots, v_n$ in this order and edges $e_0 = v_0v_1$, $e_1 = v_1v_2, \dots, e_{n-1} = v_{n-1}v_n$ is *acceptable* (*with respect to* ϕ) if e_0 is the uncolored edge and $\phi(e_i) \in \bigcup_{j<i} \bar{\phi}(v_j)$, for all $1 \leqslant i \leqslant n-1$. Kierstead proved the following theorem.

Theorem 1.1 (Kierstead [1]). *If M is a critical multigraph, $\chi'(M) > \Delta(M)+1$, and P an acceptable path with respect to a $(\chi'(M)-1)$-coloring ϕ of M, then $\bar{\phi}(v) \cap \bar{\phi}(w) = \emptyset$ for any two distinct vertices v, w of P.*

2. The main result

Theorem 2.1. *If M is a multigraph with girth g, then*

$$\chi'(M) \leqslant \Delta(M) + \left\lceil \frac{\mu(M)}{\lfloor g/2 \rfloor} \right\rceil.$$

Proof. Assume to the contrary that the statement is false. Then, there is a critical multigraph M with girth g and $\chi'(M) > \Delta(M) + \lceil \mu(M)/\lfloor g/2 \rfloor \rceil$. We may assume that there is a positive integer k such that $g = 2k+1$ or $2k$. Note, that if $g = 2k$ then $k \geqslant 2$. Henceforth, $\chi'(M) = \Delta(M) + s > \Delta(M) + \lceil \mu(M)/k \rceil$, and therefore $s \geqslant \mu(M)/k + 1$.

Let ϕ be a $(\Delta(M)+s-1)$-coloring of $M-e$, $e = x_0x_1$, and $\alpha \neq \beta$ such that $\alpha \in \bar{\phi}(x_0)$ and $\beta \in \bar{\phi}(x_1)$. Then, there is a path Q', whose edges are colored α and β alternately, with vertices $x_1, x_2, \dots, x_m, x_0$ and edges $f = x_mx_0$ and x_ix_{i+1}, for $i = 1, \dots, m-1$. Path $Q = (Q' - f) + e$ is acceptable and let P be a maximum acceptable path with initial subpath Q, i.e. $V(P) = \{x_0, x_1, \dots, x_m, x_{m+1}, \dots, x_n\}$ and $E(P) = \{x_ix_{i+1} \mid i = 0, \dots, n-1\}$.

Note that m is even, $f \in E(M)$, and $n \geqslant m \geqslant 2k$.

Let $N_P(x_n)$ be the set of neighbors of x_n in P, and

$$n - 1 = N(g-2) + r, \tag{1}$$

where $0 \leqslant r \leqslant g-3$.

Since g is the girth of G it follows that $N \geqslant 1$ and $|N_P(x_n)| \leqslant N+1$. On the other hand, P is maximum and it follows from Theorem 1.1 that there are at least $n(s-1)+2$ edges between x_n and $P - x_n$. Hence,

$$n(s-1) + 2 \leqslant |N_P(x_n)|\mu(M) \leqslant (N+1)\mu(M). \tag{2}$$

We conclude that $(n/k)\mu(M) + 2 \leqslant (N+1)\mu(M)$, and therefore $n/k < N+1$.

If $g = 2k+1$, then it follows from (1) that $(N(2k-1)+r+1)/k < N+1$, and hence $r < (k-1)(1-N)$. Since $N, k \geqslant 1$ it follows that $r < 0$, a contradiction.

If $g = 2k$, then we may assume that $k \geqslant 2$ and as above it follows from (1) that $k(N-1) < 2N - r - 1$. If $k \geqslant 3$, then $r < 2 - N$, and hence $N = 1$ and consequently $r = 0$. But then $n = 2k-1$, contradicting the fact that $n \geqslant 2k$.

If $k = 2$, then $r = 0$. Thus, $n = 2N+1$ and since m is even it follows that $n > m \geqslant 4$. Since $s \geqslant \mu(M)/2 + 1$ and $n = 2N+1$ inequality (2) implies that $|N_P(x_n)| = N+1$. This is only possible when x_n is adjacent to every vertex with even index in P, in particular

$x_n x_m \in E(M)$. But per construction of P we have $x_0 x_m \in E(P)$ and hence x_0, x_m and x_n induce a triangle in M, contradicting the fact that M is triangle-free. □

The bounds of Theorem 2.1 are attained by the following multigraphs. Let C_{2k+1}^l $(k, l \geqslant 1)$ be obtained from a circuit on $2k+1$ vertices $v_0, v_1, \dots, v_{2k}$ and with edges v_0v_1, $v_1v_2, \dots, v_{2k}v_0$. Let $\mu(v_iv_{i+1}) = \mu(C_{2k+1}^l) = lk$. The girth of C_{2k+1}^l is $g = 2k+1$, $\Delta(C_{2k+1}^l) = 2lk$ and $|E(C_{2k+1}^l)| = lk(2k+1)$. A color class can contain at most k edges, and consequently $\chi'(C_{2k+1}^l) \geqslant l(2k+1) = \Delta(C_{2k+1}^l) + l$. From Theorem 2.1 it follows that $\chi'(C_{2k+1}^l) \leqslant \Delta(C_{2k+1}^l) + l$, and hence $\chi'(C_{2k+1}^l) = \Delta(C_{2k+1}^l) + l = \Delta(C_{2k+1}^l) + \lceil \mu(C_{2k+1}^l)/\lfloor g/2 \rfloor \rceil$.

Corollary 2.2. *If M is a multigraph with girth at least $2\mu(M)$, then $\chi'(M) \leqslant \Delta(M) + 1$.*

Since every multigraph has girth at least 3, Vizing's theorem is an immediate consequence of Theorem 2.1.

Theorem 2.3 (Vizing's theorem). *If M is a multigraph, then $\chi'(M) \leqslant \Delta(M) + \mu(M)$.*

References

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